Pragmatic Programming

Session 11 - Combinatorial Game Theory and My Favorite Problem

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Combinatorial games

Combinatorial game theory concerns a very specific type of game, which must obey the following rules.

- There are two players who alternate moves.
- There exists no stochastic elements.
- Each player has perfect information.
- The game will eventually terminate.
- The game ends when the player in turn has no legal move and then that player loses.
- We will also suppose that the game is impartial, i.e., both players would have the same legal moves if they were about to play.
- We will also suppose that the game is short, i.e., only has finitely many positions.

The Example of Two Coworkers Ordering a Salad

- Two coworkers **N** and **NC** are ordering salads at the local salad place. The 1000 ingredients are laid out in a 1×1000 line. **N** and **NC** take turns picking out their ingredients, with **NC** starting since he is kind enough to let **N** get an extra stamp card for **NC**'s salad.
- There is only one problem, sometimes **N** picks red onions which **NC** despises and sometimes **NC** picks olives which **N** loathes. This difference of culinary tastes has led to an all out gastronomical dispute between the coworkers.
- Things are so bad that they will both refuse to pick an ingredient which is, or is just next to, another ingredient that has already been picked (who knows, some of the red onions/olives might have fallen into a nearby ingredient bin . . .)
- Specifically, when someone picks ingredient n they will mark $\mathtt{max}(n-1,0), n$ and $\mathtt{min}(n+1,1000)$ with an **X** with a white board marker on the glass monitor.
- Who will be the first who cannot add an extra ingredient?

Nim

The rules: On a table are n piles of sticks (represented by an array `int a[n];`). There are at most m sticks in each pile. In each move a player chooses one of the piles and removes one or more sticks from it. The player that removes the last stick wins.

- Find a dynamic programming solution. What is the space/time complexity?
- There are $\mathcal{O}(n^m)$ states in the game. For each state in the game, we have space complexity $\mathcal{O}(n)$ and time complexity that is linear in the number of legal moves: $\mathcal{O}(nm)$. Yikes.
- Find a strategy when $n \leq 2$.
- Find a strategy when $n = 3$. Is `a = { 1, 2, 3}` winning?
- In 1902, the American mathematician Charles Bouton found a solution in space complexity $\mathcal{O}(1)$ and time complexity $\mathcal{O}(n)$. Bonkers!

Solving Nim

Consider the XOR of all elements of a and call this value s. This value is either 0 or not, which sort of reflects a game when $n = 2$ and is an invariant of the game we can control.

- Consider a pile of x sticks which is reduced to y sticks ($y < x$). Then the new state will have XOR value s^+ given by

$$s^+ = s \oplus x \oplus y.$$

- If $s = 0$ then $s^+ = x \oplus y \neq 0$ since $y < x$.
- If $s \neq 0$ then if we want $s^+ = 0$ we consider

$$(s \oplus x) \oplus y = 0$$

  and solve for x, y.
- We get that $y = s \oplus x$ and that x must satisfy

$$x > s \oplus x$$

  which happens for instance if x is the pile which share the most significant set digit of s.
- By induction, this tactic is optimal.

The Grundy Value

We will call the value s the *Grundy value* of the state. This value can be extended to any impartial short game as follows. Let the Grundy value of a losing state be 0. For any non-losing state, let $\{s_i\}_{i \in I}$ be the Grundy values of all reachable states, then

$$s \coloneqq \texttt{mex}\{s_i\}_{i \in I}$$

where the function `mex` (minimum excludant) is the smallest non-negative integer not found in the given set.

- One can show that this definition of the Grundy value is consistent with the one from the last slide.
- Let a game of nim G have piles which individually form single-pile games of nim $G_1, \ldots, G_n$. If the function g assigns the game its Grundy value, we have that

$$g(G) = g(G_1) \oplus \cdots \oplus g(G_n).$$

The Sprague-Grundy Theorem

For any two impartial games G and H, we define their sum $G + H$ as the game where G and H are played in parallel, and the player about to move must choose to make a move in either of G and H.

Then the following fantastic result holds $g(G + H) = g(G) \oplus g(H)$.

- **Proof.** Let G', H' represent a move from G, H respectively. Then $s = g(G + H)$ is the `mex` of the set

$$\begin{aligned} S &\coloneqq \{g(G' + H)\}_{G'} \cup \{g(G + H')\}_{H'} \\ &\underset{\text{induction}}{=} \{g(G') \oplus g(H)\}_{G'} \cup \{g(G) \oplus g(H')\}_{H'} \end{aligned}$$

- Clearly, $g(G) \oplus g(H) \notin S$ since $g(G') \neq g(G), g(H') \neq g(H)$.
- Let $x < g(G) \oplus g(H)$ then there must exists a most significant set bit of $g(G) \oplus g(H) \oplus x$ at position k. The kth bit of $g(G) \oplus g(H)$ must be 1 and, without loss of generality, the kth bit of $g(H)$ is 1.
- So the kth bit of $g(G) \oplus x$ is 0 and any bit to the left of k in $g(G) \oplus x$ and $g(H)$ must be the same. This implies that $g(G) \oplus x < g(H)$ and that there exists some H' such that $g(H') = g(G) \oplus x \implies x = g(G) \oplus g(H')$ $\implies s = g(G) \oplus g(H)$.

The Sprague-Grundy Theorem

We say that two short impartial games G and H are equivalent if, for any short impartial game K, the games $G + K$ and $H + K$ are both either winning or losing.

Theorem: Every short impartial game is equivalent to some one-pile Nim game.

- We can now also solve the salad problem.
- Each time an **X** is drawn, the problem is split into a sum of two short impartial games.
- Therefore, the Grundy value of a salad game of length $1 \times n$ is given by

$$g(n) = \begin{cases} 0 & \text{for } n = 0, \\ 1 & \text{for } n \leq 3, \\ \texttt{mex}\left(\{g(n-2)\} \cup \{g(i-2) \oplus g(n-1-i)\}_{i=2}^{n-1}\right) & \text{otherwise.} \end{cases}$$

  and is computed in $\mathcal{O}(n)$ memory and $\mathcal{O}(n^2)$ time.

My Favorite Problem

8. The squares of an infinite chessboard are numbered as follows: in the zeroth row and column we put 0, and then in every other square we put the smallest non-negative integer that does not appear anywhere below it in the same column nor anywhere to the left of it in the same row.

...	...						
6	7	...					
5	4	7	...				
4	5	6	7	...			
3	2	1	0	7	...		
2	3	0	1	6	7	...	
1	0	3	2	5	4	7	...
0	1	2	3	4	5	6	...

What number will appear in the 2016th row and 1601st column? Can you generalize?

- This problem is the eighth out of ten problems for a thing I applied for when I was 17. In my solution, I finally ended up with the result that if we write

$$R = x_n \cdot 2^n + \ldots + x_2 \cdot 2^2 + x_1 \cdot 2 + x_0$$
$$C = y_n \cdot 2^n + \ldots + y_2 \cdot 2^2 + y_1 \cdot 2 + y_0$$

  then the value $V(R, C)$ at row R, column C in the chessboard is given by

$$V(R, C) = \sum_{i=1}^{n} |x_i - y_i| \cdot 2^i.$$

- With the Grundy values I now realize that this question can be restated as a game of nim with two piles of size R and C so that $V(R, C) = R \oplus C$.

This Week

- Skim read section 11.5 in Laaksonen[1].
- Check out *Impartial Games and Sprague-Grundy Theory — Lecture Notes* by Jonas Sjöstrand (KTH).
- Solve half of this weeks 5 problems (truncated).

[1] Antti Laaksonen. *Guide to competitive programming*. Springer, 2020.